Chapter 2
Single-Particle Motions

2.1 Introduction

What makes plasmas particularly difficult to analyze is the fact that the densities fall in an intermediate range. Fluids like water are so dense that the motions of individual molecules do not have to be considered. Collisions dominate, and the simple equations of ordinary fluid dynamics suffice. At the other extreme in very low-density devices like the alternating-gradient synchrotron, only single-particle trajectories need be considered; collective effects are often unimportant. Plasmas behave sometimes like fluids, and sometimes like a collection of individual particles. The first step in learning how to deal with this schizophrenic personality is to understand how single particles behave in electric and magnetic fields. This chapter differs from succeeding ones in that the **E** *and* **B** *fields are assumed to be prescribed* and not affected by the charged particles.

2.2 Uniform E and B Fields

2.2.1 *E = 0*

In this case, a charged particle has a simple cyclotron gyration. The equation of motion is

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B} \tag{2.1}$$

Taking $\hat{\mathbf{z}}$ to be the direction of **B** ($\mathbf{B} = B\,\hat{\mathbf{z}}$), we have

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$$m\dot{v}_x = qBv_y \quad m\dot{v}_y = -qBv_x \quad m\dot{v}_z = 0$$

$$\ddot{v}_x = \frac{qB}{m}\dot{v}_y = -\left(\frac{qB}{m}\right)^2 v_x \tag{2.2}$$

$$\ddot{v}_y = -\frac{qB}{m}\dot{v}_x = -\left(\frac{qB}{m}\right)^2 v_y$$

This describes a simple harmonic oscillator at the *cyclotron frequency*, which we define to be

$$\boxed{\omega_c \equiv \frac{|q|B}{m}} \tag{2.3}$$

By the convention we have chosen, ω_c is always nonnegative. **B** is measured in tesla, or webers/m^2, a unit equal to 10^4 G. The solution of Eq. (2.2) is then

$$v_{x,y} = v_\perp \exp(\pm i\omega_c t + i\delta_{x,y})$$

the $\pm$ denoting the sign of q. We may choose the phase δ so that

$$v_x = v_\perp e^{i\omega_c t} = \dot{x} \tag{2.4a}$$

where $v_\perp$ is a positive constant denoting the speed in the plane perpendicular to **B**. Then

$$v_y = \frac{m}{qB}\dot{v}_x = \pm\frac{1}{\omega_c}\dot{v}_x = \pm i v_\perp e^{i\omega_c t} = \dot{y} \tag{2.4b}$$

Integrating once again, we have

$$x - x_0 = -i\frac{v_\perp}{\omega_c}e^{i\omega_c t} \qquad y - y_0 = \pm\frac{v_\perp}{\omega_c}e^{i\omega_c t} \tag{2.5}$$

We define the *Larmor radius* to be

$$\boxed{r_L \equiv \frac{v_\perp}{\omega_c} = \frac{mv_\perp}{|q|B}} \tag{2.6}$$

Taking the real part of Eq. (2.5), we have

$$x - x_0 = r_L \sin\omega_c t \qquad y - y_0 = \pm r_L \cos\omega_c t \tag{2.7}$$

This describes a circular orbit around a *guiding center* (x_0, y_0) which is fixed (Fig. 2.1). The direction of the gyration is always such that the magnetic field

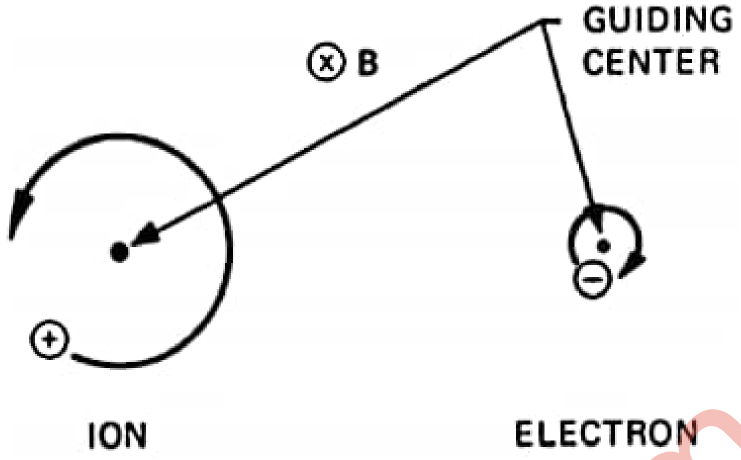


Fig. 2.1 Larmor orbits in a magnetic field

generated by the charged particle is opposite to the externally imposed field. Plasma particles, therefore, tend to *reduce* the magnetic field, and plasmas are *diamagnetic*. In Fig. 2.1, the right-hand rule with the thumb pointed in the **B** direction would give ions a clockwise gyration. Ions gyrate counterclockwise to generate an opposing **B**, thus lowering the energy of the system. In addition to this motion, there is an arbitrary velocity v_z along **B** which is not affected by **B**. The trajectory of a charged particle in space is, in general, a helix.

2.2.2 *Finite E*

If now we allow an electric field to be present, the motion will be found to be the sum of two motions: the usual circular Larmor gyration plus a drift of the guiding center. We may choose **E** to lie in the x–z plane so that $E_y = 0$. As before, the z component of velocity is unrelated to the transverse components and can be treated separately. The equation of motion is now

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{2.8}$$

whose z component is

$$\frac{dv_z}{dt} = \frac{q}{m}E_z$$

or

$$v_z = \frac{qE_z}{m}t + v_{z0} \tag{2.9}$$

This is a straightforward acceleration along **B**. The transverse components of Eq. (2.8) are

$$\frac{dv_x}{dt} = \frac{q}{m}E_x \pm \omega_c v_y$$
$$\frac{dv_y}{dt} = 0 \mp \omega_c v_x \tag{2.10}$$

Differentiating, we have (for constant **E**)

$$\ddot{v}_x = -\omega_c^2 v_x$$
$$\ddot{v}_y = \mp\omega_c\left(\frac{q}{m}E_x \pm \omega_c v_y\right) = -\omega_c^2\left(\frac{E_x}{B} + v_y\right)$$

We can write this as

$$\frac{d^2}{dt^2}\left(v_y + \frac{E_x}{B}\right) = -\omega_c^2\left(v_y + \frac{E_x}{B}\right) \tag{2.11}$$

so that Eq. (2.11) is reduced to the previous case (Eq. (2.2)) if we replace v_y by $v_y + (E_x/B)$. Equations (2.4a) and (2.4b) are therefore replaced by

$$v_x = v_\perp e^{i\omega_c t}$$
$$v_y = \pm i v_\perp e^{i\omega_c t} - \frac{E_x}{B} \tag{2.12}$$

The Larmor motion is the same as before, but there is superimposed a drift $\mathbf{v}_{gc}$ of the guiding center in the −y direction (for $E_x > 0$) (Fig. 2.2).

To obtain a general formula for $\mathbf{v}_{gc}$, we can solve Eq. (2.8) in vector form. We may omit the $m\,d\mathbf{v}/dt$ term in Eq. (2.8), since this term gives only the circular motion at ω_c, which we already know about. Then Eq. (2.8) becomes

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0 \tag{2.13}$$

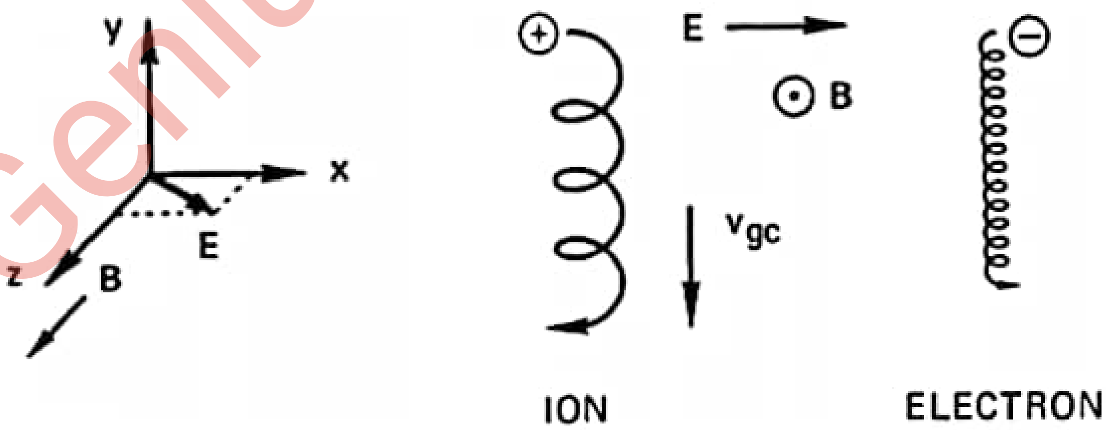


Fig. 2.2 Particle drifts in crossed electric and magnetic fields

Taking the cross product with **B**, we have

$$\mathbf{E} \times \mathbf{B} = \mathbf{B} \times (\mathbf{v} \times \mathbf{B}) = \mathbf{v}B^2 - \mathbf{B}(\mathbf{v} \cdot \mathbf{B}) \tag{2.14}$$

The transverse components of this equation are

$$\mathbf{v}_{\perp gc} = \mathbf{E} \times \mathbf{B}/B^2 \equiv \mathbf{v}_E \tag{2.15}$$

We define this to be $\mathbf{v}_E$, the electric field drift of the guiding center. In magnitude, this drift is

$$\boxed{v_E = \frac{E(\mathrm{V/m})}{B(\mathrm{tesla})} \frac{\mathrm{m}}{\mathrm{sec}}} \tag{2.16}$$

It is important to note that $\mathbf{v}_E$ is independent of q, m, and $v_\perp$. The reason is obvious from the following physical picture. In the first half-cycle of the ion's orbit in Fig. 2.2, it gains energy from the electric field and increases in $v_\perp$ and, hence, in r_L. In the second half-cycle, it loses energy and decreases in r_L. This difference in r_L on the left and right sides of the orbit causes the drift v_E. A negative electron gyrates in the opposite direction but also gains energy in the opposite direction; it ends up drifting in the same direction as an ion. For particles of the same velocity but different mass, the lighter one will have smaller r_L and hence drift less per cycle. However, its gyration frequency is also larger, and the two effects exactly cancel. Two particles of the same mass but different energy would have the same ω_c. The slower one will have smaller r_L and hence gain less energy from **E** in a half-cycle. However, for less energetic particles the fractional change in r_L for a given change in energy is larger, and these two effects cancel (Problem 2.4).

The three-dimensional orbit in space is therefore a slanted helix with changing pitch (Fig. 2.3).

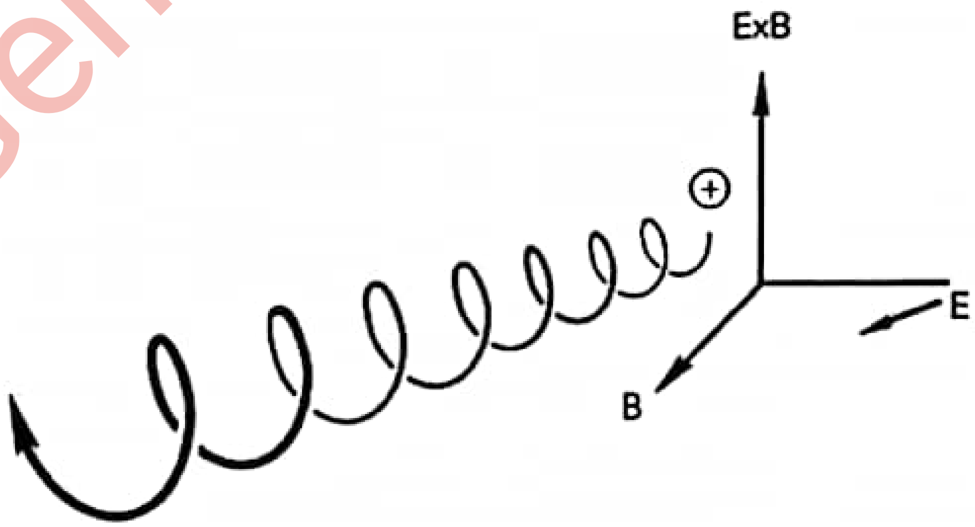


Fig. 2.3 The actual orbit of a gyrating particle in space

2.2.3 Gravitational Field

The foregoing result can be applied to other forces by replacing $q\mathbf{E}$ in the equation of motion (2.8) by a general force $\mathbf{F}$. The guiding center drift caused by $\mathbf{F}$ is then

$$\boxed{\mathbf{v}_f = \frac{1}{q}\frac{\mathbf{F} \times \mathbf{B}}{B^2}} \tag{2.17}$$

In particular, if $\mathbf{F}$ is the force of gravity $m\mathbf{g}$, there is a drift

$$\boxed{\mathbf{v}_g = \frac{m}{q}\frac{\mathbf{g} \times \mathbf{B}}{B^2}} \tag{2.18}$$

This is similar to the drift $\mathbf{v}_E$ in that it is perpendicular to both the force and $\mathbf{B}$, but it differs in one important respect. The drift $\mathbf{v}_g$ changes sign with the particle's charge. Under a gravitational force, ions and electrons drift in opposite directions, so there is a net current density in the plasma given by

$$\mathbf{j} = n(M + m)\frac{\mathbf{g} \times \mathbf{B}}{B^2} \tag{2.19}$$

The physical reason for this drift (Fig. 2.4) is again the change in Larmor radius as the particle gains and loses energy in the gravitational field. Now the electrons gyrate in the opposite sense to the ions, but the force on them is in the same direction, so the drift is in the opposite direction. The magnitude of $\mathbf{v}_g$ is usually negligible (Problem 2.6), but when the lines of force are curved, there is an effective gravitational force due to centrifugal force. This force, which is *not* negligible, is independent of mass; this is why we did not stress the m dependence of Eq. (2.18). Centrifugal force is the basis of a plasma instability called the "gravitational" instability, which has nothing to do with real gravity.

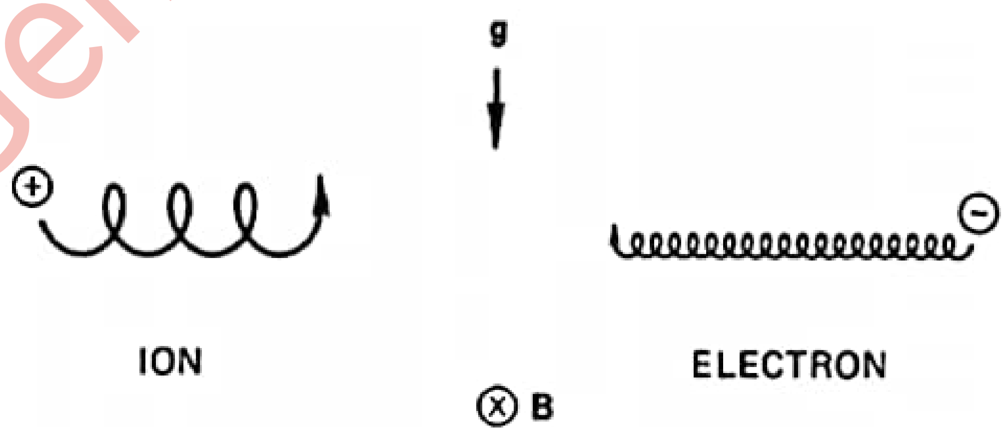


Fig. 2.4 The drift of a gyrating particle in a gravitational field

2.7. An unneutralized electron beam has density $n_e = 10^{14}\,\mathrm{m}^{-3}$ and radius $a = 1$ cm and flows along a 2-T magnetic field. If **B** is in the $+z$ direction and **E** is the electrostatic field due to the beam's charge, calculate the magnitude and direction of the $\mathbf{E} \times \mathbf{B}$ drift at $r = a$ (See Fig. P2.7).

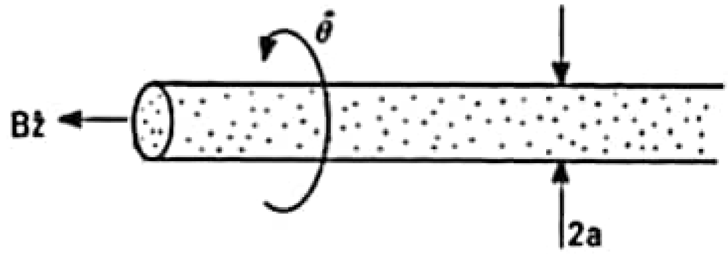


Fig. P2.7

2.3 Nonuniform B Field

Now that the concept of a guiding center drift is firmly established, we can discuss the motion of particles in inhomogeneous fields—**E** and **B** fields which vary in space or time. For uniform fields we were able to obtain exact expressions for the guiding center drifts. As soon as we introduce inhomogeneity, the problem becomes too complicated to solve exactly. To get an approximate answer, it is customary to expand in the small ratio r_L/L, where L is the scale length of the inhomogeneity. This type of theory, called *orbit theory*, can become extremely involved. We shall examine only the simplest cases, where only one inhomogeneity occurs at a time.

2.3.1 $\nabla B \perp B$: Grad-B Drift

Here the lines of force[1] are straight, but their density increases, say, in the y direction (Fig. 2.5). We can anticipate the result by using our simple physical picture. The gradient in $|B|$ causes the Larmor radius to be larger at the bottom of the orbit than at the top, and this should lead to a drift, in opposite directions for ions and electrons, perpendicular to both **B** and ∇B. The drift velocity should obviously be proportional to r_L/L and to $v_\perp$.

Consider the Lorentz force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, averaged over a gyration. Clearly, $\overline{F_x} = 0$, since the particle spends as much time moving up as down. We wish to calculate $\overline{F_y}$, in an approximate fashion, by using the *undisturbed orbit* of the particle to find the average. The undisturbed orbit is given by Eqs. (2.4a),

[1] The magnetic field lines are often called "lines of force." They are not lines of force. The misnomer is perpetuated here to prepare the student for the treacheries of his profession.

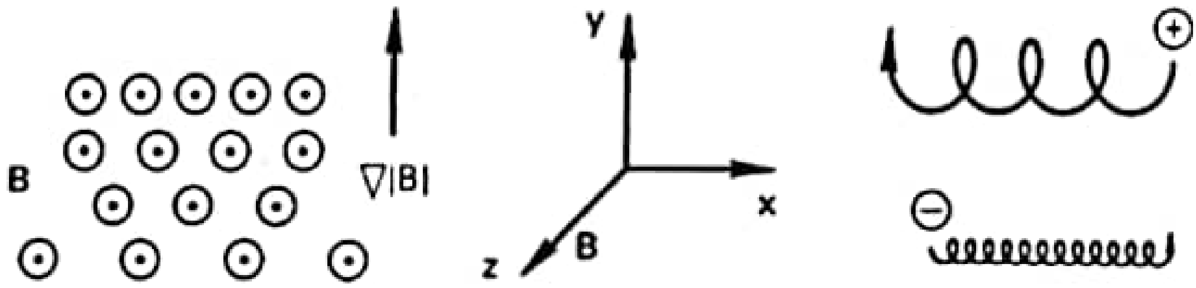


Fig. 2.5 The drift of a gyrating particle in a nonuniform magnetic field

(2.4b), and (2.7) for a uniform **B** field. Taking the real part of Eqs. (2.4a) and (2.4b), we have

$$F_y = -qv_x B_z(y) = -qv_\perp(\cos\omega_c t)\left[B_0 \pm r_L(\cos\omega_c t)\frac{\partial B}{\partial y}\right] \tag{2.20}$$

where we have made a Taylor expansion of B field about the point $x_0 = 0, y_0 = 0$ and have used Eq. (2.7):

$$\mathbf{B} = \mathbf{B}_0 + (\mathbf{r}\cdot\nabla)\mathbf{B} + \cdots \tag{2.21}$$
$$B_z = B_0 + y(\partial B_z/\partial y) + \cdots$$

This expansion of course requires $r_L/L \ll 1$, where L is the scale length of $\partial Bz/\partial y$. The first term of Eq. (2.20) averages to zero in a gyration, and the average of $\cos^2 \omega_c t$ is ½, so that

$$\overline{F}_y = \mp qv_\perp r_L \tfrac{1}{2}(\partial B/\partial y) \tag{2.22}$$

The guiding center drift velocity is then

$$\mathbf{v}_{gc} = \frac{1}{q}\frac{\mathbf{F}\times\mathbf{B}}{B^2} = \frac{1}{q}\frac{\overline{F}_y}{|B|}\hat{\mathbf{x}} = \mp\frac{v_\perp r_L}{B}\frac{1}{2}\frac{\partial B}{\partial y}\hat{\mathbf{x}} \tag{2.23}$$

where we have used Eq. (2.17). Since the choice of the y axis was arbitrary, this can be generalized to

$$\boxed{\mathbf{v}_{\nabla B} = \pm\frac{1}{2}v_\perp r_L\frac{\mathbf{B}\times\nabla B}{B^2}} \tag{2.24}$$

This has all the dependences we expected from the physical picture; only the factor ½ (arising from the averaging) was not predicted. Note that the ± stands for the sign of the charge, and lightface B stands for $|\mathbf{B}|$. The quantity $\mathbf{v}_{\nabla B}$ is called the *grad-B drift*; it is in opposite directions for ions and electrons and causes a current transverse to **B**. An exact calculation of $\mathbf{v}_{\nabla B}$ would require using the exact orbit, including the drift, in the averaging process.

2.3.2 Curved B: Curvature Drift

Here we assume the lines of force to be curved with a constant radius of curvature R_c, and we take $|B|$ to be constant (Fig. 2.6). Such a field does not obey Maxwell's equations in a vacuum, so in practice the grad-B drift will always be added to the effect derived here. A guiding center drift arises from the centrifugal force felt by the particles as they move along the field lines in their thermal motion. If $v_\parallel^2$ denotes the average square of the component of random velocity along **B**, the average centrifugal force is

$$\mathbf{F}_{cf} = \frac{mv_\parallel^2}{R_c}\hat{\mathbf{r}} = mv_\parallel^2\frac{\mathbf{R}_c}{R_c^2} \tag{2.25}$$

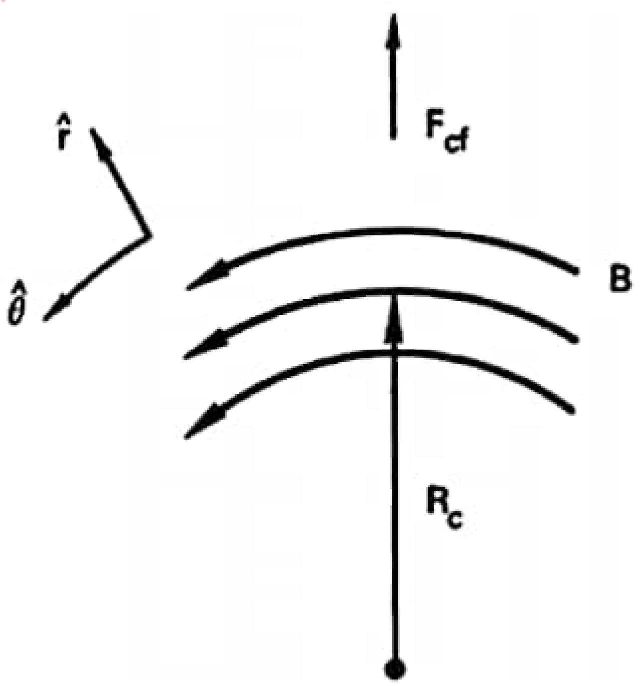
According to Eq. (2.17), this gives rise to a drift

$$\boxed{\mathbf{v}_R = \frac{1}{q}\frac{\mathbf{F}_{cf}\times\mathbf{B}}{B^2} = \frac{mv_\parallel^2}{qB^2}\frac{\mathbf{R}_c\times\mathbf{B}}{R_c^2}} \tag{2.26}$$

The drift $\mathbf{v}_R$ is called the *curvature drift*.

We must now compute the grad-B drift which accompanies this when the decrease of $|B|$ with radius is taken into account. In a vacuum, we have $\nabla\times\mathbf{B}=0$. In the cylindrical coordinates of Fig. 2.6, $\nabla\times\mathbf{B}$ has only a z component, since **B** has only a θ component and ∇B only an r component. We then have

Fig. 2.6 A curved magnetic field



$$(\nabla \times \mathbf{B})_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = 0 \qquad B_\theta \propto \frac{1}{r} \tag{2.27}$$

Thus

$$|B| \propto \frac{1}{R_c} \qquad \frac{\nabla|B|}{|B|} = -\frac{\mathbf{R}_c}{R_c^2} \tag{2.28}$$

Using Eq. (2.24), we have

$$\mathbf{v}_{\nabla B} = \mp\frac{1}{2}\frac{v_\perp r_L}{B^2}\mathbf{B} \times |B|\frac{\mathbf{R}_c}{R_c^2} = \pm\frac{1}{2}\frac{v_\perp^2}{\omega_c}\frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2 B} = \frac{1}{2}\frac{m}{q}v_\perp^2\frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2 B^2} \tag{2.29}$$

Adding this to $\mathbf{v}_R$, we have the total drift in a curved vacuum field:

$$\boxed{\mathbf{v}_R + \mathbf{v}_{\nabla B} = \frac{m}{q}\frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2 B^2}\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right)} \tag{2.30}$$

It is unfortunate that these drifts add. This means that if one bends a magnetic field into a torus for the purpose of confining a thermonuclear plasma, the particles will drift out of the torus no matter how one juggles the temperatures and magnetic fields.

For a Maxwellian distribution, Eqs. (1.7) and (1.10) indicate that $\overline{v_\parallel^2}$ and $\frac{1}{2}\overline{v_\perp^2}$ are each equal to KT/m, since $v_\perp$ involves two degrees of freedom. Equations (2.3) and (1.6) then allow us to write the average curved-field drift as

$$\overline{\mathbf{v}}_{R+\nabla B} = \pm\frac{v_{th}^2}{R_c \omega_c}\hat{\mathbf{y}} = \pm\frac{\overline{r}_L}{R_c}v_{th}\hat{\mathbf{y}} \tag{2.30a}$$

where $\hat{y}$ here is the direction of $\mathbf{R}_c \times \mathbf{B}$. This shows that $\overline{\mathbf{v}}_{R+\nabla B}$ depends on the charge of the species but not on its mass.

2.3.3 $\nabla B \| \mathbf{B}$: Magnetic Mirrors

Now we consider a magnetic field which is pointed primarily in the z direction and whose magnitude varies in the z direction. Let the field be axisymmetric, with $B_\theta = 0$ and $\partial/\partial\theta = 0$. Since the lines of force converge and diverge, there is necessarily a component B_r (Fig. 2.7). We wish to show that this gives rise to a force which can trap a particle in a magnetic field.

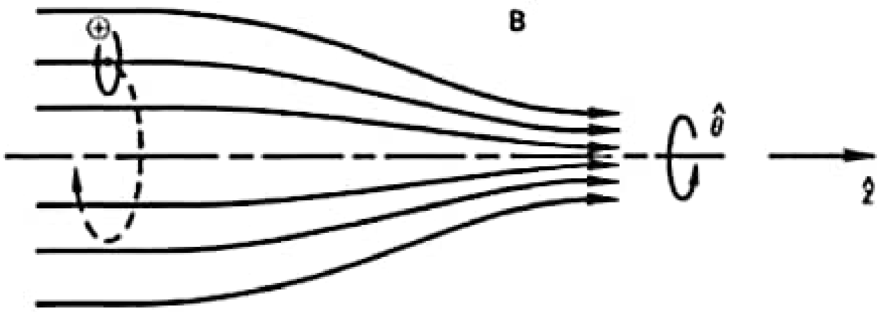


Fig. 2.7 Drift of a particle in a magnetic mirror field

We can obtain B_r from $\nabla \cdot \mathbf{B} = 0$:

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0 \tag{2.31}$$

If $\partial B_z/\partial z$ is given at $r = 0$ and does not vary much with r, we have approximately

$$\begin{aligned} rB_r &= -\int_0^r r\frac{\partial B_z}{\partial z}dr \simeq -\frac{1}{2}r^2\left[\frac{\partial B_z}{\partial z}\right]_{r=0} \\ B_r &= -\frac{1}{2}r\left[\frac{\partial B_z}{\partial z}\right]_{r=0} \end{aligned} \tag{2.32}$$

The variation of $|B|$ with r causes a grad-B drift of guiding centers about the axis of symmetry, but there is no radial grad-B drift, because $\partial B/\partial\theta = 0$. The components of the Lorentz force are

$$\begin{aligned} F_r &= q(\underset{①}{v_\theta B_z} - v_z \cancel{B_\theta}) \\ F_\theta &= q(\underset{②}{-v_r B_z} + \underset{③}{v_z B_r}) \\ F_z &= q(v_r \cancel{B_\theta} - \underset{④}{v_\theta B_r}) \end{aligned} \tag{2.33}$$

Two terms vanish if $B_\theta = 0$, and terms 1 and 2 give rise to the usual Larmor gyration. Term 3 vanishes on the axis; when it does not vanish, this azimuthal force causes a drift in the radial direction. This drift merely makes the guiding centers follow the lines of force. Term 4 is the one we are interested in. Using Eq. (2.32), we obtain

$$F_z = \frac{1}{2}qv_\theta r(\partial B_z/\partial_z) \tag{2.34}$$

We must now average over one gyration. For simplicity, consider a particle whose guiding center lies on the axis. Then v_θ is a constant during a gyration; depending on the sign of q, v_θ is $\mp v_\perp$. Since $r = r_L$, the average force is

$$\overline{F}_z = \mp \frac{1}{2} q v_\perp r_L \frac{\partial B_z}{\partial z} = \mp \frac{1}{2} q \frac{v_\perp^2}{\omega_c} \frac{\partial B_z}{\partial z} = -\frac{1}{2} \frac{m v_\perp^2}{B} \frac{\partial B_z}{\partial z} \tag{2.35}$$

We define the *magnetic moment* of the gyrating particle to be

$$\boxed{\mu \equiv \frac{1}{2} m v_\perp^2 / B} \tag{2.36}$$

so that

$$\overline{F}_z = -\mu (\partial B_z / \partial z) \tag{2.37}$$

This is a specific example of the force on a diamagnetic particle, which in general can be written

$$\mathbf{F}_\| = -\mu \partial B / \partial \mathbf{s} = -\mu \nabla_\| B \tag{2.38}$$

where ds is a line element along **B**. Note that the definition (2.36) is the same as the usual definition for the magnetic moment of a current loop with area A and current I: $\mu = IA$. In the case of a singly charged ion, I is generated by a charge e coming around $\omega_c / 2\pi$ times a second: $I = e\omega_c / 2\pi$. The area A is $\pi r_L^2 = \pi v_\perp^2 / \omega_c^2$. Thus

$$\mu = \frac{\pi v_\perp^2}{\omega_c^2} \frac{e \omega_c}{2\pi} = \frac{1}{2} \frac{v_\perp^2 e}{\omega_c} = \frac{1}{2} \frac{m v_\perp^2}{B}.$$

As the particle moves into regions of stronger or weaker **B**, its Larmor radius changes, but *μ remains invariant*. To prove this, consider the component of the equation of motion along **B**:

$$m \frac{dv_\|}{dt} = -\mu \frac{\partial B}{\partial s} \tag{2.39}$$

Multiplying by $v_\|$ on the left and its equivalent ds/dt on the right, we have

$$m v_\| \frac{dv_\|}{dt} = \frac{d}{dt}\left(\frac{1}{2} m v_\|^2\right) = -\mu \frac{\partial B}{\partial s} \frac{ds}{dt} = -\mu \frac{dB}{dt} \tag{2.40}$$

Here dB/dt is the variation of B as seen by the particle; B itself is constant. The particle's energy must be conserved, so we have

$$\frac{d}{dt}\left(\frac{1}{2}mv_{\parallel}^2+\frac{1}{2}mv_{\perp}^2\right)=\frac{d}{dt}\left(\frac{1}{2}mv_{\parallel}^2+\mu B\right)=0 \tag{2.41}$$

With Eq. (2.40) this becomes

$$-\mu\frac{dB}{dt}+\frac{d}{dt}(\mu B)=0$$

so that

$$d\mu/dt=0 \tag{2.42}$$

The invariance of μ is the basis for one of the primary schemes for plasma confinement: the *magnetic mirror*. As a particle moves from a weak-field region to a strong-field region in the course of its thermal motion, it sees an increasing B, and therefore its $v_{\perp}$ must increase in order to keep μ constant. Since its total energy must remain constant, $v_{\parallel}$ must necessarily decrease. If B is high enough in the "throat" of the mirror, $v_{\parallel}$ eventually becomes zero; and the particle is "reflected" back to the weak-field region. It is, of course, the force $\mathbf{F}_{\parallel}$ which causes the reflection. The nonuniform field of a simple pair of coils forms two magnetic mirrors between which a plasma can be trapped (Fig. 2.8). This effect works on both ions and electrons.

The trapping is not perfect, however. For instance, a particle with $v_{\perp}=0$ will have no magnetic moment and will not feel any force along $\mathbf{B}$. A particle with small $v_{\perp}/v_{\parallel}$ at the midplane ($B=B_0$) will also escape if the maximum field B_m is not large enough. For given B_0 and B_m, which particles will escape? A particle with $v_{\perp}=v_{\perp 0}$ and $v_{\parallel}=v_{\parallel 0}$ at the midplane will have $v_{\perp}=v'_{\perp}$ and $v_{\parallel}=0$ at its turning point. Let the field be B' there. Then the invariance of μ yields

$$\tfrac{1}{2}mv_{\perp 0}^2/B_0=\tfrac{1}{2}mv_{\perp}'^2/B' \tag{2.43}$$

Conservation of energy requires

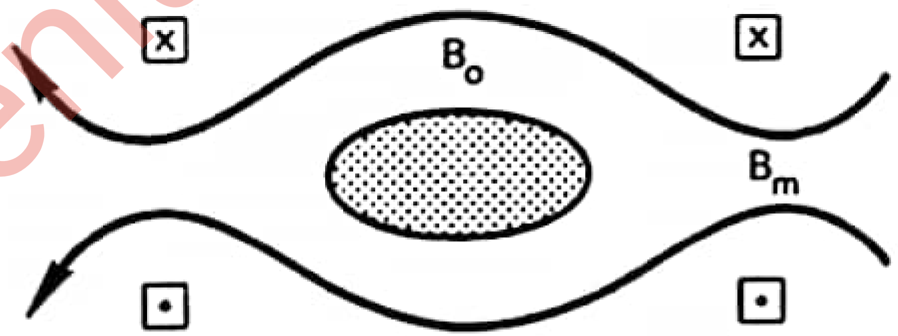


Fig. 2.8 A plasma trapped between magnetic mirrors

$$v_\perp'^2 = v_{\perp 0}^2 + v_{\parallel 0}^2 \equiv v_0^2 \tag{2.44}$$

Combining Eqs. (2.43) and (2.44), we find

$$\frac{B_0}{B'} = \frac{v_{\perp 0}^2}{v_\perp'^2} = \frac{v_{\perp 0}^2}{v_0^2} \equiv \sin^2\theta \tag{2.45}$$

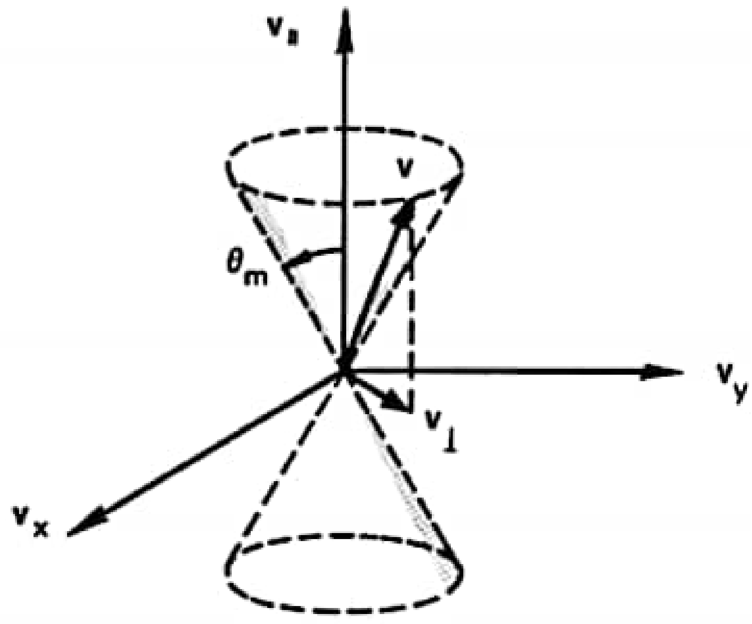
where θ is the pitch angle of the orbit in the weak-field region. Particles with smaller θ will mirror in regions of higher B. If θ is too small, B' exceeds B_m; and the particle does not mirror at all. Replacing B' by B_m in Eq. (2.45), we see that the smallest θ of a confined particle is given by

$$\sin^2\theta_m = B_0/B_m \equiv 1/R_m \tag{2.46}$$

where R_m is the *mirror ratio*. Equation (2.46) defines the boundary of a region in velocity space in the shape of a cone, called a *loss cone* (Fig. 2.9). Particles lying within the loss cone are not confined. Consequently, a mirror-confined plasma is never isotropic. Note that the loss cone is independent of q or m. Without collisions, both ions and electrons are equally well confined. When collisions occur, particles are lost when they change their pitch angle in a collision and are scattered into the loss cone. Generally, electrons are lost more easily because they have a higher collision frequency.

The magnetic mirror was first proposed by Enrico Fermi as a mechanism for the acceleration of cosmic rays. Protons bouncing between magnetic mirrors approaching each other at high velocity could gain energy at each bounce (Fig. 2.10). How such mirrors could arise is another story. A further example of the mirror effect is the confinement of particles in the Van Allen belts. The magnetic field of the earth, being strong at the poles and weak at the equator, forms a natural mirror with rather large R_m.

Fig. 2.9 The loss cone



2.4 Nonuniform E Field

Now we let the magnetic field be uniform and the electric field be nonuniform. For simplicity, we assume **E** to be in the x direction and to vary sinusoidally in the x direction (Fig. 2.11):

$$\mathbf{E} \equiv E_0(\cos kx)\hat{\mathbf{x}} \tag{2.47}$$

This field distribution has a wavelength $\lambda = 2\pi/k$ and is the result of a sinusoidal distribution of charges, which we need not specify. In practice, such a charge distribution can arise in a plasma during a wave motion. The equation of motion is

$$m(d\mathbf{v}/dt) = q[\mathbf{E}(x) + \mathbf{v} \times \mathbf{B}] \tag{2.48}$$

whose transverse components are

$$\dot{v}_x = \frac{qB}{m}v_y + \frac{q}{m}E_x(x) \qquad \dot{v}_y = -\frac{qB}{m}v_x \tag{2.49}$$

$$\ddot{v}_x = -\omega_c^2 v_x \pm \omega_c \frac{\dot{E}_x}{B} \tag{2.50}$$

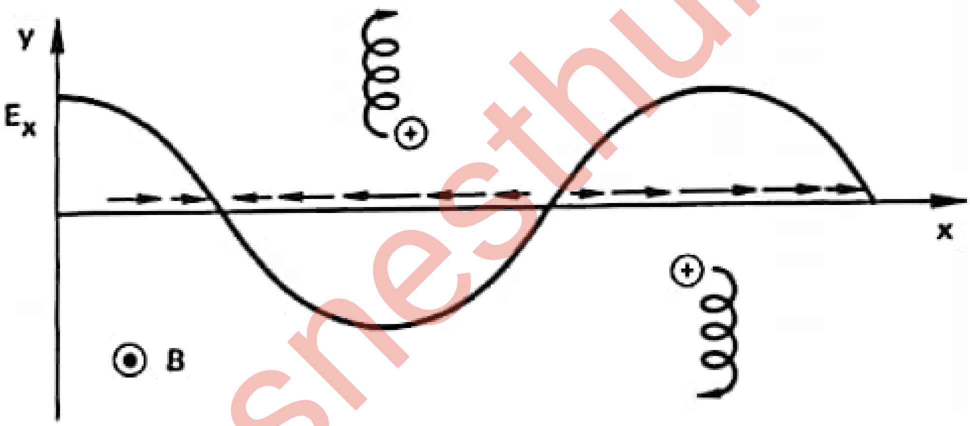


Fig. 2.11 Drift of a gyrating particle in a nonuniform electric field

$$\ddot{v}_y = -\omega_c^2 v_y - \omega_c^2 \frac{E_x(x)}{B} \tag{2.51}$$

Here $E_x(x)$ is the electric field at the position of the particle. To evaluate this, we need to know the particle's orbit, which we are trying to solve for in the first place. If the electric field is weak, we may, as an approximation, use the *undisturbed orbit* to evaluate $E_x(x)$. The orbit in the absence of the E field was given in Eq. (2.7):

$$\ddot{v}_y = -\omega_c^2 v_y - \omega_c^2 \frac{E_x(x)}{B} \tag{2.51}$$

Here $E_x(x)$ is the electric field at the position of the particle. To evaluate this, we need to know the particle's orbit, which we are trying to solve for in the first place. If the electric field is weak, we may, as an approximation, use the *undisturbed orbit* to evaluate $E_x(x)$. The orbit in the absence of the E field was given in Eq. (2.7):

$$x = x_0 + r_L \sin \omega_c t \tag{2.52}$$

From Eqs. (2.51) and (2.47), we now have

$$\ddot{v}_y = -\omega_c^2 v_y - \omega_c^2 \frac{E_0}{B} \cos k(x_0 + r_L \sin w_c t) \tag{2.53}$$

Anticipating the result, we look for a solution which is the sum of a gyration at ω_c and a steady drift v_E. Since we are interested in finding an expression for v_E, we take out the gyratory motion by averaging over a cycle. Equation (2.50) then gives $\bar{v}_x = 0$. In Eq. (2.53), the oscillating term $\ddot{v}_y$ clearly averages to zero, and we have

$$\bar{\ddot{v}}_y = 0 = -\omega_c^2 \bar{v}_y - \omega_c^2 \frac{E_0}{B} \overline{\cos k(x_0 + r_L \sin \omega_c t)} \tag{2.54}$$

Expanding the cosine, we have

$$\begin{aligned} \cos k(x_0 + r_L \sin \omega_c t) = {} & \cos (kx_0) \cos (kr_L \sin \omega_c t) \\ & - \sin (kx_0) \sin (kr_L \sin \omega_c t) \end{aligned} \tag{2.55}$$

It will suffice to treat the small Larmor radius case, $kr_L \ll 1$. The Taylor expansions

$$\begin{aligned} \cos \varepsilon &= 1 - \tfrac{1}{2}\varepsilon^2 + \cdots \\ \sin \varepsilon &= \varepsilon + \cdots \end{aligned} \tag{2.56}$$

allow us to write

$$\cos k(x_0 + r_L \sin \omega_c t) \approx (\cos kx_0)\left(1 - \tfrac{1}{2} k^2 r_L^2 \sin^2 \omega_c t\right) - (\sin kx_0) k r_L \sin \omega_c t$$

The last term vanishes upon averaging over time, and Eq. (2.54) gives

$$\bar{v}_y = -\frac{E_0}{B} (\cos kx_0)\left(1 - \tfrac{1}{4} k^2 r_L^2\right) = -\frac{E_x(x_0)}{B}\left(1 - \tfrac{1}{4} k^2 r_L^2\right) \tag{2.57}$$

Thus the usual $\mathbf{E} \times \mathbf{B}$ drift is modified by the inhomogeneity to read

$$\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2}\left(1 - \frac{1}{4}k^2 r_L^2\right) \tag{2.58}$$

The physical reason for this is easy to see. An ion with its guiding center at a maximum of E actually spends a good deal of its time in regions of weaker E. Its average drift, therefore, is less than E/B evaluated at the guiding center. In a linearly varying E field, the ion would be in a stronger field on one side of the orbit and in a field weaker by the same amount on the other side; the correction to $\mathbf{v}_E$ then cancels out. From this it is clear that the correction term depends on the *second derivative* of E. For the sinusoidal distribution we assumed, the second derivative is always negative with respect to E. For an arbitrary variation of E, we need only replace ik by ∇ and write Eq. (2.58) as

$$\boxed{\mathbf{v}_E = \left(1 + \frac{1}{4}r_L^2 \nabla^2\right)\frac{\mathbf{E} \times \mathbf{B}}{B^2}} \tag{2.59}$$

The second term is called the *finite-Larmor-radius effect*. What is the significance of this correction? Since r_L is much larger for ions than for electrons, $\mathbf{v}_E$ is no longer independent of species. If a density clump occurs in a plasma, an electric field can cause the ions and electrons to separate, generating another electric field. If there is a feedback mechanism that causes the second electric field to enhance the first one, E grows indefinitely, and the plasma is unstable. Such an instability, called a *drift instability*, will be discussed in a later chapter. The grad-B drift, of course, is also a finite-Larmor-radius effect and also causes charges to separate. According to Eq. (2.24), however, $\mathbf{v}_{\nabla B}$ is proportional to kr_L, whereas the correction term in Eq. (2.58) is proportional to $k^2 r_L^2$. The nonuniform-E-field effect, therefore, is important at relatively large k, or small scale lengths of the inhomogeneity. For this reason, drift instabilities belong to a more general class called *microinstabilities*.

2.5 Time-Varying E Field

Let us now take E and B to be uniform in space but varying in time. First, consider the case in which E alone varies sinusoidally in time, and let it lie along the x axis:

$$\mathbf{E} = E_0 e^{i\omega t}\hat{\mathbf{x}} \tag{2.60}$$

Since $\dot{E}_x = i\omega E_x$, we can write Eq. (2.50) as

$$\ddot{v}_x = -\omega_c^2\left(v_x \mp \frac{i\omega}{\omega_c}\frac{\tilde{E}_x}{B}\right) \tag{2.61}$$

Let us define

$$\tilde{v}_p \equiv \pm \frac{i\omega}{\omega_c} \frac{\tilde{E}_x}{B}$$
$$\tilde{v}_E \equiv \frac{\tilde{E}_x}{B} \tag{2.62}$$

where the tilde has been added merely to emphasize that the drift is oscillating. The upper (lower) sign, as usual, denotes positive (negative) q. Now Eqs. (2.50) and (2.51) become

$$\ddot{v}_x = -\omega_c^2 \left(v_x - \tilde{v}_p\right)$$
$$\ddot{v}_y = -\omega_c^2 \left(v_y - \tilde{v}_E\right) \tag{2.63}$$

By analogy with Eq. (2.12), we try a solution which is the sum of a drift and a gyratory motion:

$$v_x = v_\perp e^{i\omega_c t} + \tilde{v}_p$$
$$v_y = \pm i v_\perp e^{i\omega_c t} + \tilde{v}_E \tag{2.64}$$

If we now differentiate twice with respect to time, we find

$$\ddot{v}_x = -\omega_c^2 v_x + \left(\omega_c^2 - \omega^2\right)\tilde{v}_p$$
$$\ddot{v}_y = -\omega_c^2 v_y + \left(\omega_c^2 - \omega^2\right)\tilde{v}_E \tag{2.65}$$

This is not the same as Eq. (2.63) unless $\omega^2 \ll \omega_c^2$. If we now make the assumption that E varies slowly, so that $\omega^2 \ll \omega_c^2$, then Eq. (2.64) is the approximate solution to Eq. (2.63).

Equation (2.64) tells us that the guiding center motion has two components. The y component, perpendicular to **B** and **E**, is the usual $\mathbf{E} \times \mathbf{B}$ drift, except that v_E now oscillates slowly at the frequency ω. The x component, a new drift *along the direction of* **E**, is called the *polarization drift*. By replacing $i\omega$ by $\partial/\partial t$, we can generalize Eq. (2.62) and define the polarization drift as

$$\boxed{\mathbf{v}_p = \pm \frac{1}{\omega_c B} \frac{d\mathbf{E}}{dt}} \tag{2.66}$$

Since $\mathbf{v}_p$ is in opposite directions for ions and electrons, there is a *polarization current*; for $Z = 1$, this is

$$\mathbf{j}_p = ne\left(\mathbf{v}_{ip} - \mathbf{v}_{ep}\right) = \frac{ne}{eB^2}(M + m)\frac{d\mathbf{E}}{dt} = \frac{\rho}{B^2}\frac{d\mathbf{E}}{dt} \tag{2.67}$$

where ρ is the mass density.

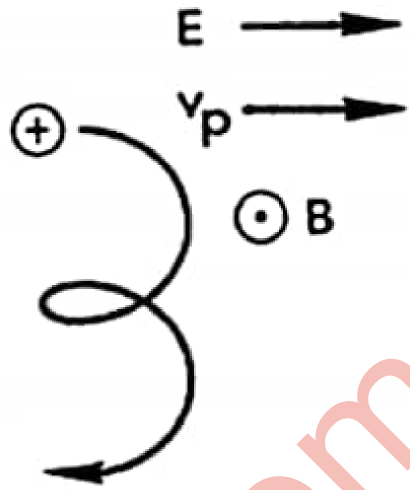


Fig. 2.12 The polarization drift

The physical reason for the polarization current is simple (Fig. 2.12). Consider an ion at rest in a magnetic field. If a field **E** is suddenly applied, the first thing the ion does is to move in the direction of **E**. Only after picking up a velocity **v** does the ion feel a Lorentz force $e\mathbf{v} \times \mathbf{B}$ and begin to move downward in Fig. (2.12). If **E** is now kept constant, there is no further $\mathbf{v}_p$ drift but only a $\mathbf{v}_E$ drift. However, if **E** is reversed, there is again a momentary drift, this time to the left. Thus $\mathbf{v}_p$ is a startup drift due to inertia and occurs only in the first half-cycle of each gyration during which **E** changes. Consequently, $\mathbf{v}_p$ goes to zero with ω/ω_c.

The polarization effect in a plasma is similar to that in a solid dielectric, where $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$. The dipoles in a plasma are ions and electrons separated by a distance r_L. But since ions and electrons can move around to preserve quasineutrality, the application of a steady **E** field does not result in a polarization field **P**. However, if **E** oscillates, an oscillating current $\mathbf{j}_p$ results from the lag due to the ion inertia.

2.6 Time-Varying B Field

Finally, we allow the magnetic field to vary in time. Since the Lorentz force is always perpendicular to **v**, a magnetic field itself cannot impart energy to a charged particle. However, associated with **B** is an electric field given by

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}} \tag{2.68}$$

and this can accelerate the particles. We can no longer assume the fields to be completely uniform. Let $\mathbf{v}_\perp = d\mathbf{l}/dt$ be the transverse velocity, **l** being the element of path along a particle trajectory (with $v_\parallel$ neglected). Taking the scalar product of the equation of motion (2.8) with $\mathbf{v}_\perp$, we have

$$\frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) = q\mathbf{E} \cdot \mathbf{v}_\perp = q\mathbf{E} \cdot \frac{d\mathbf{l}}{dt} \tag{2.69}$$

The change in one gyration is obtained by integrating over one period:

$$\delta\left(\tfrac{1}{2}mv_\perp^2\right) = \int_0^{2\pi/\omega_c} q\mathbf{E}\cdot\frac{d\mathbf{l}}{dt}dt$$

If the field changes slowly, we can replace the time integral by a line integral over the unperturbed orbit:

$$\begin{aligned}\delta\left(\tfrac{1}{2}mv_\perp^2\right) &= \oint q\mathbf{E}\cdot d\mathbf{l} = q\int_s (\nabla\times\mathbf{E})\cdot d\mathbf{S} \\ &= -q\int_S \dot{\mathbf{B}}\cdot d\mathbf{S}\end{aligned} \tag{2.70}$$

Here **S** is the surface enclosed by the Larmor orbit and has a direction given by the right-hand rule when the fingers point in the direction of **v**. Since the plasma is diamagnetic, we have $\mathbf{B}\cdot d\mathbf{S} < 0$ for ions and >0 for electrons. Then Eq. (2.70) becomes

$$\delta\left(\tfrac{1}{2}mv_\perp^2\right) = \pm q\dot{B}\pi r_L^2 = \pm q\pi\dot{B}\frac{v_\perp^2}{\omega_c}\frac{m}{\pm qB} = \frac{\frac{1}{2}mv_\perp^2}{B}\cdot\frac{2\pi\dot{B}}{\omega_c} \tag{2.71}$$

The quantity $2\pi\dot{B}/\omega_c = \dot{B}/f_c$ is just the change δB during one period of gyration. Thus

$$\delta\left(\tfrac{1}{2}mv_\perp^2\right) = \mu\,\delta B \tag{2.72}$$

Since the left-hand side is $\delta(\mu B)$, we have the desired result

$$\delta\mu = 0 \tag{2.73}$$

The magnetic moment is invariant in slowly varying magnetic fields.

As the B field varies in strength, the Larmor orbits expand and contract, and the particles lose and gain transverse energy. This exchange of energy between the particles and the field is described very simply by Eq. (2.73). The invariance of μ allows us to prove easily the following well-known theorem:

The magnetic flux through a Larmor orbit is constant.

The flux Φ is given by BS, with $S = \pi r_L^2$. Thus

$$\Phi = B\pi\frac{v_\perp^2}{\omega_c^2} = B\pi\frac{v_\perp^2 m^2}{q^2B^2} = \frac{2\pi m}{q^2}\frac{\frac{1}{2}mv_\perp^2}{B} = \frac{2\pi m}{q^2}\mu \tag{2.74}$$

Therefore, Φ is constant if μ is constant.

This property is used in a method of plasma heating known as *adiabatic compression*. Figure 2.13 shows a schematic of how this is done. A plasma is injected into the region between the mirrors A and B. Coils A and B are then pulsed

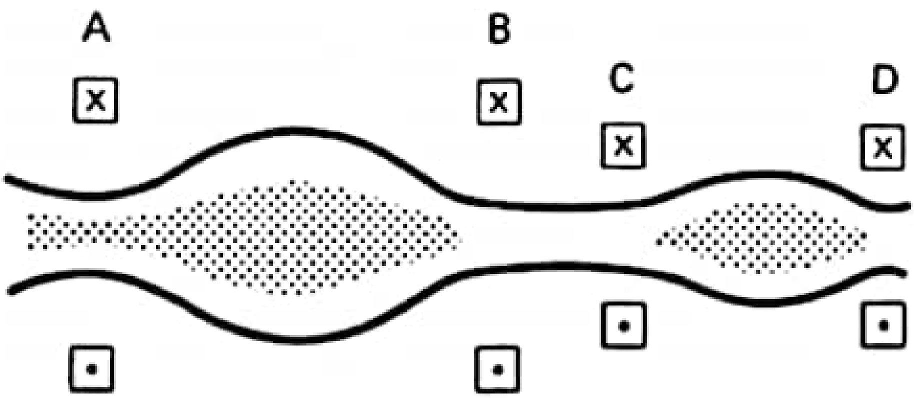


Fig. 2.13 Two-stage adiabatic compression of a plasma

to increase B and hence $v_\perp^2$. The heated plasma can then be transferred to the region C–D by a further pulse in A, increasing the mirror ratio there. The coils C and D are then pulsed to further compress and heat the plasma. Early magnetic mirror fusion devices employed this type of heating. Adiabatic compression has also been used successfully on toroidal plasmas and is an essential element of laser-driven fusion schemes using either magnetic or inertial confinement.

2.7 Summary of Guiding Center Drifts

General force $\mathbf{F}$: $$\mathbf{v}_f = \frac{1}{q}\frac{\mathbf{F}\times\mathbf{B}}{B^2} \quad (2.17)$$

Electric field : $$\mathbf{v}_E = \frac{\mathbf{E}\times\mathbf{B}}{B^2} \quad (2.15)$$

Gravitational field : $$\mathbf{v}_g = \frac{m}{q}\frac{\mathbf{g}\times\mathbf{B}}{B^2} \quad (2.18)$$

Nonuniform **E** : $$\mathbf{v}_E = \left(1 + \frac{1}{4}r_L^2\nabla^2\right)\frac{\mathbf{E}\times\mathbf{B}}{B^2} \quad (2.59)$$

Nonuniform **B** *field*

Grad – B drift : $$\mathbf{v}_{\nabla B} = \pm\frac{1}{2}v_\perp r_L \frac{\mathbf{B}\times\nabla B}{B^2} \quad (2.24)$$

Curvature drift : $$\mathbf{v}_R = \frac{mv_\parallel^2}{q}\frac{\mathbf{R}_c\times\mathbf{B}}{R_c^2 B^2} \quad (2.26)$$

$$\textit{Curved vacuum field:} \quad \mathbf{v}_R + \mathbf{v}_{\nabla B} = \frac{m}{q}\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right)\frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2 B^2} \tag{2.30}$$

$$\textit{Polarization drift:} \quad \mathbf{v}_p = \pm\frac{1}{\omega_c B}\frac{d\mathbf{E}}{dt} \tag{2.66}$$

2.8 Adiabatic Invariants

It is well known in classical mechanics that whenever a system has a periodic motion, the action integral $\oint p\,dq$ taken over a period is a constant of the motion. Here p and q are the generalized momentum and coordinate which repeat themselves in the motion. If a slow change is made in the system, so that the motion is not quite periodic, the constant of the motion does not change and is then called an *adiabatic invariant*. By slow here we mean slow compared with the period of motion, so that the integral $\oint p\,dq$ is well defined even though it is strictly no longer an integral over a closed path. Adiabatic invariants play an important role in plasma physics; they allow us to obtain simple answers in many instances involving complicated motions. There are three adiabatic invariants, each corresponding to a different type of periodic motion.

2.8.1 *The First Adiabatic Invariant,* μ

We have already met the quantity

$$\mu = mv_\perp^2/2B$$

and have proved its invariance in spatially and temporally varying **B** fields. The periodic motion involved, of course, is the Larmor gyration. If we take p to be angular momentum $mv_\perp r$ and dq to be the coordinate $d\theta$, the action integral becomes

$$\oint p\,dq = \oint mv_\perp r_L\,d\theta = 2\pi r_L m v_\perp = 2\pi\frac{mv_\perp^2}{\omega_c} = 4\pi\frac{m}{|q|}\mu \tag{2.75}$$

Thus μ is a constant of the motion as long as q/m is not changed. We have proved the invariance of μ only with the implicit assumption $\omega/\omega_c \ll 1$, where ω is a frequency characterizing the rate of change of **B** as seen by the particle. A proof exists, however, that μ is invariant even when $\omega \leq \omega_c$. In theorists' language, μ is invariant "to *all* orders in an expansion in ω/ω_c." What this means in practice is that μ remains much more nearly constant than **B** does during one period of gyration.