RATE EQUATION

Rate eq, tells us about how much electrons are entering or leaving an energy level.

RATE EQUATION FOR TWO LEVEL SYSTEM:

We know that two level laser not exist, So there would be no rate equation. But we need to prove it mathematically via equations.

(i) Stimulated Absorption:-

Number of atoms involving in this process are:

$$N_{ab} = B_{12}\rho_{\nu_0} N_1 \Delta t \qquad \therefore W_{12} = B_{12}\rho_{\nu_0}$$

$$\boxed{N_{ab} = W_{12} N_1}$$

→ Δt was also there in equation, but we are ignoring for time being because it will eventually cancelout at end.

(ii) Spontaneous Emission:-

This process involves two types of decays (Radiative and non-radiative)

$$N_{sp} = T_{21} N_2 \Delta t$$

$$\boxed{N_{sp} = T_{21} N_2}$$

$$N_{sp} = (A_{21} + S_{21})\rho_{\nu_0} N_2$$

Radiative Decay
EM wave emitted during transition

Non-Radiative Decay
No EM wave emitted, decay due to heat or collision.

(iii) Stimulated Emission:

Number of atoms involving in this process are given by

$$N_{st} = B_{21}\rho_{\nu_0} N_2 \Delta t$$

$$\boxed{N_{st} = W_{21} N_2}$$

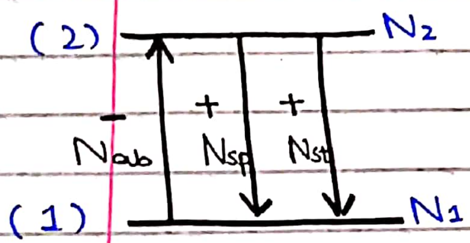
- **Rate of change of population of LEVEL-1 is:**

$$\frac{dN_1}{dt} = \underbrace{-W_{12}N_1}_{\text{going from } 1\to2} \underbrace{+T_{21}N_2}_{\text{going from } 2\to1} \underbrace{+W_{21}N_2}_{\text{going from } 2\to1} \quad \rightarrow (1)$$

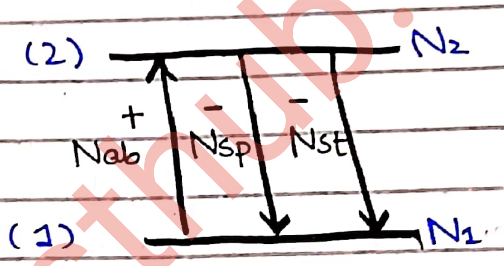
- **Rate of change of population of LEVEL-2 is:**

$$\frac{dN_2}{dt} = \underbrace{N_{ab}}_{\text{Absorption}} - \underbrace{N_{sp}}_{\text{spontaneous}} - \underbrace{N_{st}}_{\text{stimulated}}$$

$$'' = \underbrace{+W_{12}N_1}_{\text{going from } 1\to2} \underbrace{-T_{21}N_2}_{\text{from } 2\to1} \underbrace{-W_{21}N_2}_{\text{going from } 2\to1} \quad \rightarrow (2)$$



Towards N_1



Away from N_2

STEADY STATE CONDITION:

The electrons present in system are excited to N_2 and again deexcited to N_1 level. Hence, no new electrons are entering in the system and system is said to be in "Steady state Condition".

$$\frac{d}{dt}(N_1+N_2)=0 \;;\; \frac{dN_1}{dt}=0 \;;\; \frac{dN_2}{dt}=0.$$

→ In Steady state condition, the number of electrons entering at level is equal to that leaving that level at specific time. Hence, the overall change in population of two level is zero.

eq (1) and eq (2) written as

$$\frac{dN_1}{dt} = W(N_2 - N_1) + T_{21} N_2 \rightarrow (3)$$

$$\frac{dN_2}{dt} = W(N_1 - N_2) - T_{21} N_2 \rightarrow (4)$$

Total Rate of change.

$$\boxed{\frac{d(N_1 + N_2)}{dt} = 0}$$

Add eq (3) and (4) we get

$$\cancel{W(N_1 - N_2)} - \cancel{T_{21} N_2} - \cancel{W(N_2 - N_1)} + \cancel{T_{21} N_2} = 0$$

(iii) Number of atoms entering or leaving are same that's why laser is not achieved.

* Two level is not possible / steady state condition cannot exist in actual $N_1 >> N_2$.

From eq (1)
$dN_1/dt = 0$
$W_{12} = W_{21} = W$
Rate of st. absor equal to rate of st-emission.

$$\frac{dN_1}{dt} = -W_{12}N_1 + W_{21}N_2 + T_{21}N_2$$

$$0 = -W_{12}N_1 + T_{21}N_1 + W_{21}N_2$$

$$-WN_2 = -WN_1 + T_{21}N_1$$

$$N_2 = N_1 - \frac{T_{21}N_1}{W}$$

$$\downarrow \frac{N_2}{N_1} \uparrow = \frac{W - T_{21}}{W} \downarrow \uparrow \quad N_1 > N_2$$

Or you can do it this way

$$\frac{dN_2}{dt} = W_{12}N_1 - W_{21}N_2 - T_{21}N_2$$

$$0 = WN_1 - WN_2 - T_{21}N_2$$

$$0 = W(N_1 - N_2) - T_{21}N_2 \quad \text{"OR"}$$

$$0 = WN_1 - N_2(W + T_{21})$$

$$N_2(W + T_{21}) = WN_1$$

$$\downarrow \frac{N_2}{N_1} \uparrow = \frac{W}{W + T_{21}} \downarrow \uparrow \qquad \boxed{N_1 > N_2}$$

From the rate eq it is clear that population of level 1 would always be greater than level-2 while in real two level laser doesn't exist.

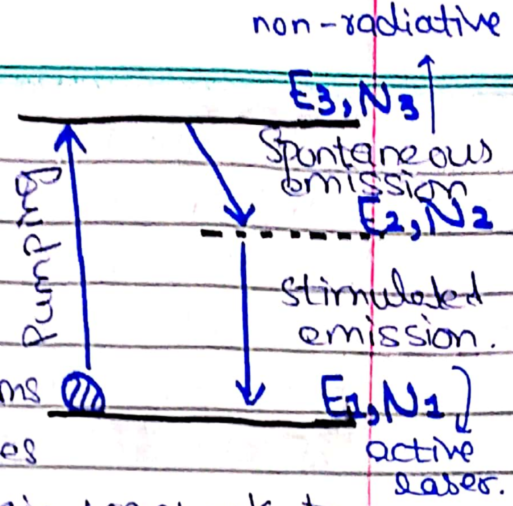
Conclusion

RATE EQUATION FOR THREE LEVEL LASER SYSTEM:-

Electrons in level-1 would pump directly to level-3 and after completing its life time it decays spontaneously through "radiative and non-radiative decay".

Radiative → A

Non-Radiative → S

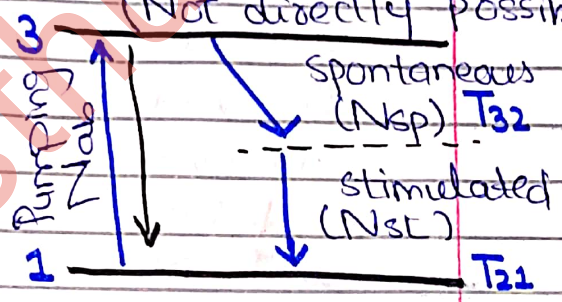


Spontaneously emitted photons interact with atoms in level-2 and stimulates them. The electron again comeback to it's ground state level by radiative decay. The rate equation for each level is written as:-

(i) For LEVEL-3 :-

$$\frac{dN_3}{dt} = + W_{13}N_1 - W_{31}N_3 - T_{32}N_3$$

towards 3, 1 → 3; away from 3, 3 → 1 (Not directly possible); away from 3, 3 → 2



*Pumping rate is same for 1 → 3 or 3 → 1

$W_{13} = W_{31} = W$

So,

$$\frac{dN_3}{dt} = W_{13}N_1 - W_{31}N_3 - T_{32}N_3$$

$$\boxed{\text{''} = W_p(N_1 - N_3) - T_{32}N_3} \rightarrow (1)$$

(ii) For LEVEL-2 :-

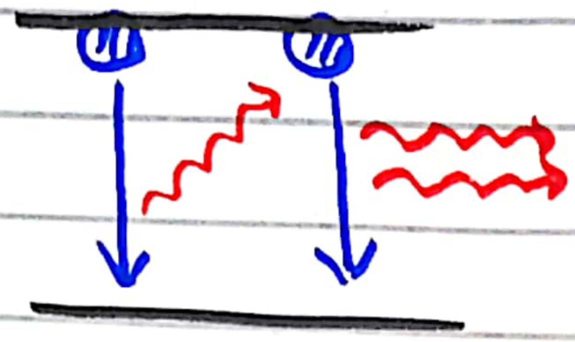
The rate equation is given as:-

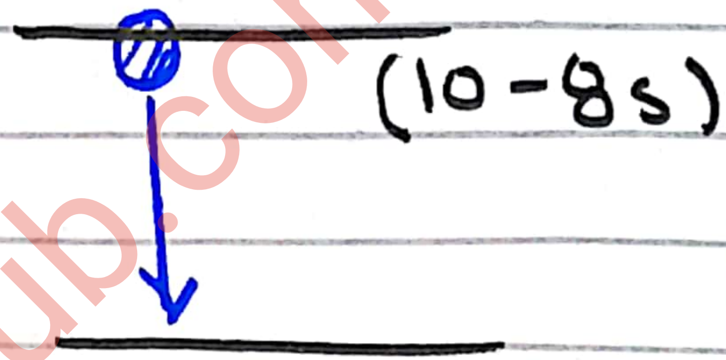
$$\frac{dN_2}{dt} = + W_{12}N_1 - W_{21}N_2 + T_{32}N_3 - T_{21}N_2$$

towards 2, 1 → 2; away from, 2 → 1; towards 2, 3 → 2; away from 2, 2 → 1

$$\boxed{\frac{dN_2}{dt} = W_{12}(N_1 - N_2) + T_{32}N_3 - T_{21}N_2} \rightarrow (2)$$

Spontaneous Emission are of two types





One in which a photon excite other electron and it come down spontaneously.

The second is that in which electron comes down itself after completing its lifetime

(iii) **For LEVEL - 1 :-**

The population is the sum of that coming from 2 and 3 level.

The population that is subtracted from level 2 and 3 is being added up to level - 1

$$\frac{dN_1}{dt} = \underbrace{(+) W_{31}N_3}_{\text{towards 1, } 3\to1} \; \underbrace{(-) W_{13}N_1}_{1\to3} \; \underbrace{(+) W_{21}N_2}_{2\to1} \; \underbrace{(-) W_{12}N_1}_{1\to2}$$

$$\underbrace{(+) T_{21}N_2}_{\text{from } 2\to1}$$

Pumping rate is same from $1\to3$ or from $3\to1$

$$W_{13} = W_{31} = W_p$$

Similarly, $W_{12} = W_{21}$

$$\boxed{\frac{dN_1}{dt} = W_p(N_3 - N_1) + W_{12}(N_2 - N_1) + T_{21}N_2} \rightarrow (3)$$

→ **At STEADY STATE :-**

The rate of change of level - 2 is equal to the rate of change of level - 1, equal to level - 3 which is equal to zero

$$\frac{dN_1}{dt} = \frac{dN_2}{dt} = \frac{dN_3}{dt} = 0$$

From eq (1) :- $\frac{dN_3}{dt} = W_p(N_1 - N_3) - T_{32}N_3$

We are trying to find seperate population of 3 and 2 levels in terms of level 1 to find total we write

$N = N_1 + N_2 + N_3$.

$$0 = W_p(N_1 - N_3) - T_{32}N_3$$

$$T_{32}N_3 = W_pN_1 - W_pN_3$$

$$T_{32}N_3 + W_pN_3 = W_pN_1$$

$$N_3(T_{32} + W_p) = W_pN_1$$

$$\boxed{N_3 = \frac{W_p(N_1)}{(T_{32} + W_p)}} \rightarrow (5)$$

in terms of N_1 and N_3

From eq (2):-

$$\frac{dN_2}{dt} = W_{12}(N_1 - N_2) + T_{32}N_3 - T_{21}N_2$$

$$0 = W_{12}N_1 - W_{12}N_2 + T_{32}\left(\frac{W_pN_1}{T_{32}+W_p}\right) - T_{21}N_2$$

$$0 = W_{12}N_1 - N_2(W_{12}+T_{21}) + T_{32}\left(\frac{W_pN_1}{T_{32}+W_p}\right)$$

$$N_2(W_{12}+T_{21}) = W_{12}N_1 + T_{32}\left(\frac{W_pN_1}{T_{32}+W_p}\right)$$

$$'' \quad = \frac{W_{12}N_1T_{32} + W_{12}N_1W_p + T_{31}W_pN_1}{T_{32}+W_p}$$

$$N_2(W_{12}+T_{21}) = \frac{N_1(W_{12}T_{32} + W_{12}W_p + T_{32}W_p)}{(T_{32}+W_p)}$$

$$\boxed{N_2 = \frac{N_1(W_{12}T_{32} + W_{12}W_p + T_{32}W_p)}{(W_{12}+T_{21})(T_{32}+W_p)}} \rightarrow (6)$$

N_2 in terms of N_1

* Now we have to take difference of population b/w three levels in which lasing action take place.

$$\boxed{\Delta N = N_1 - N_2}$$

$$\Delta N = \frac{N_1(W_{12}T_{32} + W_{12}W_p + T_{32}W_p)}{(W_{12}+T_{21})(T_{32}+W_p)} - N_1$$

$$'' = N_1\left[\frac{W_{12}T_{32} + W_{21}W_p + T_{32}W_p}{W_{12}T_{32} + W_{12}W_p + T_{32}T_{21} + W_pT_{21}} - 1\right]$$

$$\frac{\Delta N}{N_1} = \frac{W_{12}T_{32} + W_{12}W_p + T_{32}W_p - W_{12}T_{32} - W_{12}W_p - T_{32}T_{21} - W_pT_{21}}{(W_{12}+T_{21})(T_{32}+W_p)}$$

$$\Delta N = \frac{W_p (T_{32} - T_{21}) - T_{32} T_{21}}{(W_{12} + T_{31})(T_{32} + W_p)} \cdot N_1 \rightarrow (7)$$

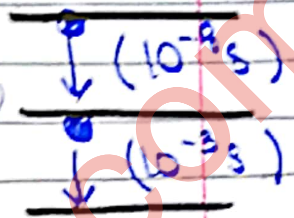
• Conditions for minimum PUMPING RATE:-

(1) $T_{32} > T_{21}$ "or" $\tau_{21} > \tau_{32}$

$T_{32} \rightarrow$ Relaxation rate for level-3

$T_{21} \rightarrow$ Relaxation rate for level-2

* $T = \frac{1}{\tau}$ Relaxation rates are inversely proportional to the lifetimes.



* To achieve Pumping minimum (Threshold Condition)

$N_2 = N_1 \longrightarrow N_2 - N_1 = 0 = \Delta N$

From eq (7)

$$[W_p (T_{32} - T_{21}) - T_{32} T_{21}] N_1 = 0$$

$$W_p (T_{32} - T_{21}) = T_{32} T_{21}$$

$$\boxed{W_p = \frac{T_{32} T_{21}}{T_{32} - T_{21}}} \rightarrow (8)$$

Pumping Rate in terms of LIFE TIME

(2) If $T_{32} >> T_{21}$ then $W_p \simeq T_{21}$

$$W_p = \frac{T_{32} T_{21}}{T_{32} - T_{21}} \qquad \because T_{21} \rightarrow 0$$

$$W_p = \frac{\cancel{T_{32}} T_{21}}{\cancel{T_{32}}}$$

$$\boxed{W_p \simeq T_{21}}$$

Condition for minimum pumping

→ **Total Population:-**

$$N = N_1 + N_2 + N_3$$

$$N = N_1 + N_1\frac{(W_{12}W_p + W_{12}T_{32} + W_pT_{32})}{(W_p+T_{32})(W_{12}+T_{21})} + \frac{N_1 W_p}{W_p+T_{32}}$$

$$N = N_1\left[1 + \frac{W_{12}W_p + W_{12}T_{32} + W_pT_{32}}{(W_p+T_{32})(W_{12}+T_{21})} + \frac{W_p}{W_p+T_{32}}\right]$$

$$'' = N_1\left[\frac{W_pW_{12} + W_pT_{21} + T_{32}W_{12} + T_{32}T_{21} + W_pW_{12} + W_{12}T_{32} + W_pT_{32} + W_p(T_{21}+T_{12})}{(W_p+T_{32})(W_{12}+T_{21})}\right]$$

$$N = N_1\frac{\left[3W_pW_{12} + 2W_pT_{21} + 2W_{12}T_{32} + W_pT_{32} + T_{32}T_{21}\right]}{(T_{32}+W_p)(W_{12}+T_{21})} \quad (9)$$

As eq (7) and eq (9) have same denominator so we can divide (7) by (9) to get change in population by total population.

→ **Change in Population by total population:-**

$$\frac{\Delta N}{N} = \frac{W_p(T_{32}-T_{21}) - T_{32}T_{21}}{3W_pW_{12} + 2W_pT_{21} + 2W_{12}T_{32} + W_pT_{32} + T_{32}T_{21}}$$

* Divide equation by T_{32}

$$\frac{\Delta N}{N} = \frac{W_p\cancel{T_{32}}/\cancel{T_{32}} - W_pT_{21}/T_{32} - \cancel{T_{32}}T_{21}/\cancel{T_{32}}}{W_p + T_{21} + (3W_p\cancel{W_{12}}^{0} + \frac{2}{T_{32}}W_p\cancel{T_{21}}^{0} + W_{12}\cancel{T_{32}}^{0})}$$

$$\frac{\Delta N}{N} = \frac{W_p\left(1 - \frac{T_{21}}{T_{32}} - T_{21}\right)}{W_p + T_{21}}$$

Condition:-

Atoms going from $1 \to 3$ are greater than that going from $2 \to 1$

$W_p >> W_{12}$; $W_{12} \approx 0$

Relaxation rate of level-3 is far greater than level-2

$T_{32} >> T_{21}$; $T_{21} \approx 0$

$$\boxed{\frac{\Delta N}{N} = \frac{W_p - T_{21}}{W_p + T_{21}}} \rightarrow (10)$$

* In order to solve the laser system, we use some approximations/conditions

Now, suppose if the population of level-3 is very small ($N_3 \to 0$)

from eq (10)

$$\frac{\Delta N}{N} = \frac{N_2 - N_1}{N_1 + N_2 + N_3} = \frac{N_2 - N_1}{N_1 + N_2} = \frac{W_p - T_{21}}{W_p + T_{21}} = \frac{\Delta N}{N}$$

$$\frac{N_2 - N_1}{N_1 + N_2} = \frac{W_p - T_{21}}{W_p + T_{21}}$$

$$N_2 W_p + N_2 T_{21} - N_1 W_p - N_1 T_{21} = N_1 W_p + N_2 W_p - T_{21} N_1 - T_{21} N_2$$

$$N_2 T_{21} + N_2 T_{21} = N_1 W_p + N_1 W_p$$

$$\cancel{2} N_2 T_{21} = \cancel{2} N_1 W_p$$

$$W_p = \frac{T_{21} N_2}{N_1} \rightarrow (11)$$

From eq (11) it tells us about the pumping rate of atoms being pumped from level 1 to level-2. At threshold level ($N_1 = N_2$)

$$W_p = T_{21}$$

Here T_{21} = spontaneous emission rate from $2 \rightarrow 1$ or T_{21} = Relaxation rate per unit time per unit volume.

PUMPING POWER:

$$P = W_p N_1 h\nu_p$$

ν_p = frequency of atoms

$\therefore W_p = T_{21}$

$$P = T_{21} N_1 h\nu_p$$

Condition:

If $N_1 = N_2$ and $N_3 \rightarrow 0$ then

$N = N_1 + N_2 + N_3 = N_1 + N_2 = N_1 + N_1 = 2N_1$

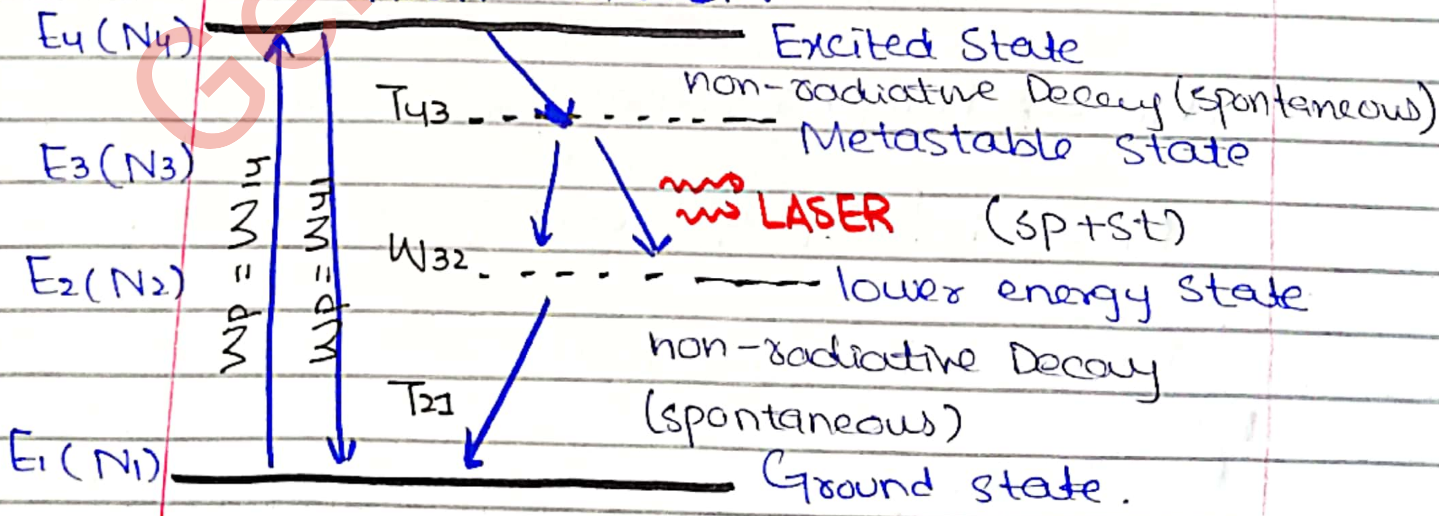
So, $\therefore N_1 = N/2$

$$P = T_{21}(N/2) h\nu_p$$

$$P = (1/\tau_{21})\left(\frac{N h\nu_p}{2}\right)$$

$$\boxed{P = \frac{N h\nu_p}{2\tau_{21}}}$$ Pumping power

RATE EQUATION FOR FOUR LEVEL LASER SYSTEM:



In four level lasers there are four levels (ground, lower, metastable and excited level)

E_1:- Pumping occurs from level $1 \rightarrow 4$ during absorption. The rate at which population increases in level - 4 during absorption is equal to rate at which it decreases (during emission).

E_2 and E_4:- has lower lifetime compared to E_3.

E_3:- is the metastable state having large lifetime.

* Lasing occurs in level 2 and 3

Rate eq for E_4:-

$$\frac{dN_4}{dt} = +W_{14}N_1 - W_{41}N_4 - T_{43}N_4$$

towards 4

$1 \rightarrow 4$

away from 4

$4 \rightarrow 1$

Spontaneous emission rate away from 4

$4 \rightarrow 3$

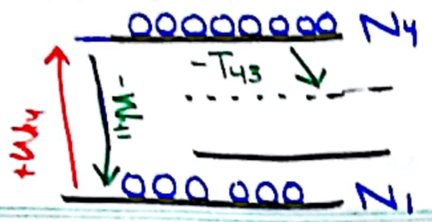
$$\frac{dN_4}{dt} = W_pN_1 - W_pN_4 - T_{43}N_4$$

$$= W_p(N_1 - N_4) - T_{43}N_4 \rightarrow (1)$$

$W_p \rightarrow$ Pumping rate for $1 \rightarrow 4$

$R_1 \rightarrow$ Pumping rate for level 2

$R_2 \rightarrow$ Pumping rate for level 3



Rate eq for E_3:-

$$\frac{dN_3}{dt} = R_2 - W_{32}N_3 + W_{23}N_2 - T_{32}N_3$$

towards 3, $2 \rightarrow 3$

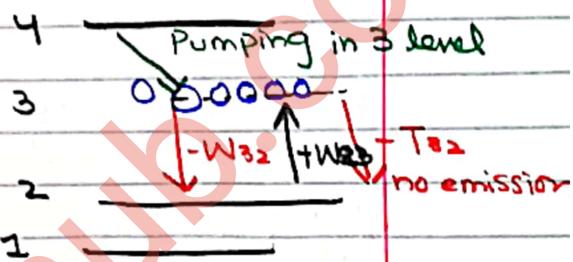
Stimulated Em. away from 3, $3 \rightarrow 2$

Spontaneous Em. $3 \rightarrow 2$ away from 3

* $W_{23} = W_{32}$

Pumping rate is same for 3 and 2

$$\frac{dN_3}{dt} = R_3 - W_{32}(N_3 - N_2) - T_{32}N_3 \rightarrow (2)$$



Rate eq for LEVEL-2:-

$$\frac{dN_2}{dt} = R_1 + W_{32}N_2 - W_{23}N_2 + T_{32}N_3 - T_{21}N_2$$

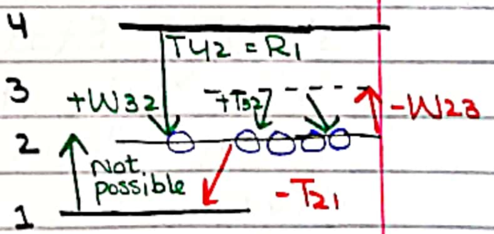
$3 \rightarrow 2$ towards 2

$2 \rightarrow 3$ away from 2

$3 \rightarrow 2$ towards 2

$2 \rightarrow 1$ away from 2

$$\frac{dN_1}{dt} = R_1 + W_{32}(N_3 - N_2) + T_{32}N_3 - T_{21}N_1 \rightarrow (3)$$

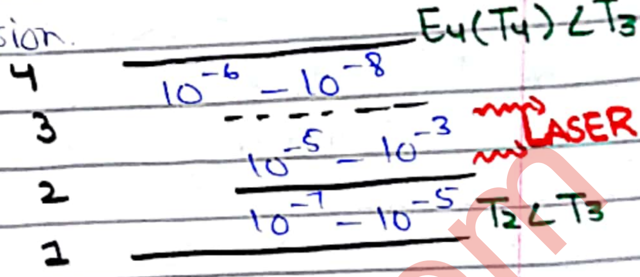


level 2 would be pumped by level-4 and 3 but not by level 1 because the thermal energy of level 1 is less than level-2

$E_2 >> KT = E_1$ ∴ KT is the ground level energy

Through optical pumping, atoms can move to any of the upper levels but they cannot exceed their thermal energy. This is the reason why level 1 atoms cannot excite to level-2

$\boxed{\text{Also}}$ the lifetime of E_2 must be smaller than E_3 in order to achieve population inversion.



Rate eq for LEVEL 1:-

away from 1

$1 \rightarrow 4$

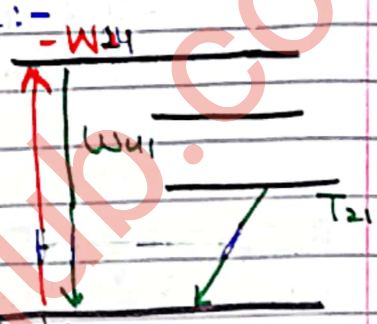
$$\frac{dN_1}{dt} = +W_{41}N_4 - W_{14}N_1 + T_{21}N_2$$

towards 1

$4 \rightarrow 1$

towards 1

$2 \rightarrow 1$



$$\frac{dN_1}{dt} = W_p(N_4 - N_1) + T_{21}N_2 \rightarrow (4)$$

Steady State Condition:-

The state of change of population of all level is same.

$$\frac{dN_4}{dt} = \frac{dN_3}{dt} = \frac{dN_2}{dt} = \frac{dN_1}{dt} = 0 \rightarrow (5)$$

* Lasering action occurs between 2 and 3 level so, using 2 and 3 (adding both we get)

$$\frac{dN_2}{dt} = \frac{dN_1}{dt} = 0$$

$$R_2 - W_{32}(N_3 - N_2) - T_{32}N_3 = 0$$

$$R_1 + W_{32}(N_3 - N_2) + T_{32}N_3 - T_{21}N_2 = 0$$

$$R_1 + R_2 - T_{21} N_2 = 0 \rightarrow (6)$$

Using Condition:-

Optical pumping in R_1 is much smaller than R_2 $\quad R_1 >> R_2$; $R_1 = 0$

$$\cancel{R_1}^{0} + R_2 - T_{21} N_2 = 0$$

$$\boxed{N_2 = \frac{R_2}{T_{21}}} \rightarrow (7)$$

From eq (2)

$$R_2 - W_{32}(N_3 - N_2) - T_{32} N_3 = 0$$

$$R_2 = W_{32}(N_3 - N_2) + T_{32} N_3$$

$$R_2 = W_{32} N_3 - W_{32} N_2 + T_{32} N_3$$

$$" = N_3 (W_{32} + T_{32}) - W_{32} N_2$$

$$R_2 + W_{32}\left(\frac{R_2}{T_{21}}\right) = N_3 (W_{32} + T_{32})$$

$$R_2\left(1 + \frac{W_{32}}{T_{21}}\right) = N_3 (W_{32} + T_{32})$$

$$R_2\left(1 + \frac{W_{32}}{T_{21}}\right) \times \frac{1}{W_{32} + T_{32}} = N_3$$

Change in Population:-

$$\Delta N = N_3 - N_2$$

$$\Delta N = \left[R_2\left(1 + \frac{W_{32}}{T_{21}}\right) \times \frac{1}{W_{32} + T_{32}}\right] - \frac{R_2}{T_{21}}$$

$$" = R_2\left[\left(\frac{T_{21} + W_{32}}{T_{21}}\right)\left(\frac{1}{W_{32} + T_{32}}\right) - \frac{1}{T_{21}}\right]$$

$$" = R_2\left[\frac{T_{21} + W_{32}}{T_{21}(T_{32} + W_{32})} - \frac{1}{T_{21}}\right]$$

$$" = R_2\left[\frac{T_{21} + \cancel{W_{32}} - T_{32} - \cancel{W_{32}}}{T_{21}(T_{32} + W_{32})}\right]$$

$$\Delta N = R_2\left[\frac{T_{21} - T_{32}}{T_{21}(T_{32} + W_{32})}\right] = N_3 - N_2$$

Change in Population in terms of spontaneous emission.

Condition:-

(1) Spontaneous terms consist of both radiative and non-radiative terms. If the radiative terms is really small, then

$$T_{21} = A_{21} + S_{21}^{\nearrow 0} \simeq A_{21}$$

$$T_{32} = A_{32} + S_{32}^{\nearrow 0} \simeq A_{32}$$

As we know, $T = 1/\tau \rightarrow A = 1/\tau$

$$A_{21} = \frac{1}{\tau_{21}} \quad ; \quad A_{32} = 1/\tau_{32}$$

As the non-radiative spontaneous decay from level $2 \rightarrow 1$ is greater than $3 \rightarrow 2$ so, the lifetime from $2 \rightarrow 1$ is smaller than $3 \rightarrow 2$

$$A_{21} >> A_{32} \quad ; \quad \tau_{21} >> \tau_{32}$$

* In 4-level lasers, if $W_{32} = 0$,

$\Delta N =$ +ive value

$$\Delta N = R_2 \left[\frac{T_{21} - T_{32}}{T_{21}(T_{32} + W_{21})} \right] = \frac{R_2 (T_{21} - T_{32})}{T_{21} T_{32}}$$

* whereas in 3-level laser if $W_{32} = 0$

$$\Delta N = N_3 - N_2 = \text{-ive value}$$